Relaxation Phenomena in Spin and Harmonic Oscillator Systems

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A method is developed for generating relaxation by introducing a fundamental interval τ and a stirring hypothesis. The application to spin and harmonic oscillator systems is discussed in some detail. All the results are obtained by exact calculations without applying perturbation theory as the systems considered are simple and completely soluble. Equations similar to phenomenological Bloch equations are derived in the case of spin systems. The relaxation times obtained by the application of the theory are not only proportional to the strength of interaction, but also to the fundamental interval τ which plays an important role in the theory. It is shown that in the case of a harmonic oscillator system, an initial Boltzmann distribution relaxes to a final equilibrium Boltzman distribution through a sequence of transient Boltzman distributions.

INTRODUCTION

THIS is a continuation of our work on stochastic dynamics of quantum-mechanical systems.[1] We have shown in our previous work that non-Hamiltonian dynamics is essential for relaxing systems without going into the details of generating relaxation mechanisms. Here we develop a theory for generating relaxation and apply it to simple systems such as coupled spins and coupled harmonic oscillators.

The unitary time development of the density matrix is characteristic of all isolated physical systems and others which are acted upon by external agencies but do not react back on them. An example of the latter type is an ensemble of particles with spin (and associated magnetic moment) in an external magnetic field. On the other hand, one may be interested in a physical system A which forms part of a larger system or is in interaction with another system B; for instance, in paramagnetic relaxation, a system of spins in interaction with lattice. Again, in a system with several degrees of freedom, one may be interested in just one or a few of these; for example, the spin state of particles scattered by a target possessing spin, irrespective of the distribution of their directions. In all these cases it is possible to define density matrices which refer only to the system or degree of freedom of interest and thus contain just the desired information. Such a density matrix, say, ρ^A may be obtained by taking a partial trace of the density matrix ρ^{AB} describing the complete interacting system. Thus, if A denotes a complete set of commuting observables for the system A of interest and B the additional set of observables required to specify the state of the total system $(A+B)$ completely, one can choose a representation of the density matrix ρ^{AB} labeled by the eigenvalues a, b of A, B. Then ρ^A has matrix elements defined in terms ρ^{AB} by

$$\rho_{aa'}{}^A = \rho_{ab,a'b}{}^{AB} \tag{1.1}$$

with the density matrix ρ^A satisfying the following conditions[2-4]:

$$(\phi, \rho^A \psi) = (\rho^A \phi, \psi) \quad \text{(hermiticity)}, \tag{1.2}$$

$$(\phi, \rho^A \phi) \geqslant 0 \quad \text{(positive-semidefiniteness)}, \tag{1.3}$$

$$\mathrm{Tr}\rho^A = 1 \quad \text{(normalization)}. \tag{1.4}$$

[In (1.1) and throughout we have invoked the summation convention whenever repeated indices occur.] While the development of $\rho^{AB}(t)$ is unitary that of $\rho^A(t)$ is not unitary in general. The specification of ρ^A at one time t_1 is not sufficient to determine ρ^A at any future time t_2. For this, in general, one has to refer back to the equation for ρ^{AB}, solve it to obtain $\rho^{AB}(t_2)$ and then evaluate $\rho^A(t_2)$ by using (1.1). However, there are situations of great physical interest where one can write down an equation for ρ^A that gives very close approximation to the actual time development of ρ^A and thus avoid the necessity for appealing to ρ^{AB} at every stage. The classic example of this is the Bloch equations[5] for paramagnetic relaxation, in which the effect of the interaction of spin (A) with reservoir (B) are lumped into a few relaxation times.

In Sec. 2 we develop a general program for relaxing systems by introducing a "stirring hypothesis" with a fundamental interval τ. In Secs. 3, 4, 5, and 6 we discuss in some detail relaxation mechanism of two coupled spins and derive equations similar to Bloch equations. Finally, in Sec. 7 we treat an example of a coupled harmonic oscillator and show that an initial Boltzmann distribution relaxes to a final equilibrium Boltzmann distribution through a sequence of transient Boltzmann distributions.

2. GENERAL PROGRAM FOR RELAXING SYSTEMS

We consider a system C composed of two subsystems A and B: A corresponds to the "physical system" of

[1] E. C. G. Sudarshan, P. M. Mathews, and Jayaseetha Rau, Phys. Rev. **121**, 920 (1961).

[2] J. Von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).

[3] R. C. Tolman, *The Principles of Statistical Mechanics* (Clarendon Press, Oxford, 1938).

[4] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).

[5] F. Bloch, Phys. Rev. **70**, 460 (1946).

interest; B corresponds to the "reservoir" or "thermal bath." The composite system $AB=C$ is considered as a complete system even though B is kept in an inexhaustible temperature bath. We can, of course, include "external forces" in the Hamiltonian of the composite system. Now we write the density matrix for the complete system C in the following fashion:

$$\rho^C = \rho^{AB} \Rightarrow \rho_{ab,a'b'}{}^C, \qquad (2.1)$$

where the indices a, a' correspond to the "coordinates" of the subsystem A and similarly b, b' correspond to the coordinates of the subsystem B. Now we consider a special class of density matrices whose dependence of b, b' is simply of the form $\delta_{bb'}$. Often the density matrix for the complete system is given by

$$\rho_{ab,a'b'}{}^C = \rho_{aa'}{}^A P^B \delta_{bb'}, \qquad (2.2)$$

where ρ^A is the density matrix for the subsystem A and $P^B \delta_{bb'}$ is the density matrix for the subsystem B.

Similarly, a general Hamiltonian for the combined system is given by

$$\mathcal{3C}^C \to \mathcal{3C}_{ab,a'b'} = \mathcal{3C}^A + \mathcal{3C}^B + \mathcal{3C}_{\text{int}}, \qquad (2.3)$$

where $\mathcal{3C}^A$ and $\mathcal{3C}^B$ are the Hamiltonians of the systems A and B, respectively, and $\mathcal{3C}_{\text{int}}$ is the interaction energy operator of the combined system.

Now let us suppose we are interested in finding how the system A relaxes when it is in contact with system B. We first start with a density matrix of the subsystem A which is $\rho_{aa'}{}^A$ and "extend" it to be a density matrix for the combined system C by defining

$$\rho_{ab,a'b'}{}^C = \rho_{aa'}{}^A P^B \delta_{bb'}.$$

A similar process of "extension" can be defined for Hamiltonians of systems A and B also:

$$\mathcal{3C}_{aa'}{}^A \to \mathcal{3C}_{ab,a'b'}{}^C \equiv \mathcal{3C}_{aa'}{}^A \delta_{bb'}, \qquad (2.4)$$

$$\mathcal{3C}_{bb'}{}^B \to \mathcal{3C}_{ab,a'b'}{}^C \equiv \mathcal{3C}_{bb'}{}^B \delta_{aa'}. \qquad (2.5)$$

The Hamiltonians $\mathcal{3C}^A$ and $\mathcal{3C}^B$ correspond to energy operators of the systems A and B, respectively, when the systems are not in contact with each other. If the systems when brought into contact do not interact then no relaxation takes place. Interaction between the two systems A and B is the mechanism which brings about the relaxation phenomena. Therefore, to produce relaxation, the general Hamiltonian must contain in addition to such extended subsystems Hamiltonians (2.4) and (2.5), a true two-system Hamiltonian ("interaction Hamiltonian") which can not be written as the sum of two independent terms.

Finally, we define a process of "stirring" which associates with every general density matrix of the combined system $C=AB$, a special density matrix C by the following correspondence $\mathcal{C}$:

$$\rho_{ab,a'b'}{}^C \Rightarrow \mathcal{C}\{\rho_{ab,a'b'}{}^C\} = \rho_{an,a'n}{}^C \rho_{bb'}{}^T. \qquad (2.6)$$

This process of stirring thus defines for every density matrix ρ^C a special density matrix $\mathcal{C}(\rho^C)$; this correspondence is linear but, in general, not unitary; in fact as can be seen it has no inverse since many general matrices are mapped into a special density matrix.

This process also associates with every general density matrix ρ^C a density matrix ρ^A of the subsystem A by the way of "restriction":

$$\rho_{ab,a'b'}{}^C \Rightarrow \rho_{aa'}{}^A = \rho_{a\beta,a'\beta}{}^C \qquad (2.7)$$

and the mapping $\mathcal{C}(\rho_{ab,a'b'}{}^C)$ is simply the "extension" of this "restriction."

Now let us start with the density matrix of the subsystem of interest $\rho_{aa'}{}^A(0)$ at time $t=0$, and extend it to $\rho_{ab,a'b'}(0) = \rho_{an,a'n}{}^A \rho_{bb'}{}^\tau$ and compute the density matrix $\rho^C(1\tau)$ at τ by the following unitary transformation:

$$\rho^C(1\tau) = U^C(\tau)\rho^C(0)U^{C\dagger}(\tau), \qquad (2.8)$$

where $U^C(\tau) = \exp(i\tau\mathcal{3C}^C)$ and $\mathcal{3C}^C$ is an appropriate combined system Hamiltonian containing an "interaction." The quantity τ is an appropriate parameter which is intrinsic to the system which we have referred as "fundamental interval" previously.

Now we compute a new density matrix $\rho^A(1\tau)$ for the subsystem A given by

$$\rho_{aa'}{}^A(1\tau) = (U^C(\tau)\rho^C(0)U^{C\dagger})_{a\beta,a'\beta}. \qquad (2.10)$$

We extend the system $\rho_{aa'}{}^A(1\tau)$ to $\rho_{ab,a'b'}{}^C(1\tau)$ by the following relation:

$$\rho_{ab,a'b'}{}^C(1\tau) = \rho_{an,a'n}{}^A(1\tau)\rho_{bb'}{}^T. \qquad (2.11)$$

We again compute $U^C(1\tau)\rho^C(1\tau)U^{C\dagger}(1\tau)$ and obtain $\rho^C(2\tau)$. By the same procedure we obtain a new density matrix for the subsystem of interest $\rho_{aa'}{}^A(2\tau)$. We continue this process n times and obtain

$$\rho_{aa'}{}^A(n\tau) = \{U^C(\tau)\rho^C[(n-1)\tau]U^{C\dagger}(\tau)\}_{a\beta,a'\beta}, \qquad (2.12)$$

$$\rho_{ab,a'b'}{}^C[(n-1)\tau] = \rho_{an,a'n}{}^A[(n-1)\tau]\rho_{bb'}{}^T. \qquad (2.13)$$

The integer-valued variable n now serves as a "time parameter" and, in general, as n increases the subsystem density matrix $\rho^A(n\tau)$ corresponds to more and more relaxation. We may call the nth case an n-step relaxation.

The following assumptions have been made in the formulation of generating relaxation developed above.

(I) Statistical independence of the density matrix of system A and system B at $t=0$, or, in other words, the density matrix of the combined system could be factorized into density matrices of component systems at $t=0$, i.e., $\rho^C(0) = \rho^A(0)\rho^B(0)$.

(II) The relaxation times (e.g., T_1 and T_2 for spin-$\frac{1}{2}$ systems) of the system of interest, A, are much larger than the fundamental interval τ.

(III) Time averages are never taken but at every fundamental interval τ, partial trace with respect to the reservoir is taken.

(IV) The system B is assumed to be in contact with an inexhaustible reservoir so that the change in the system B is negligible. At every interval τ the system B goes back to its original state at $t=0$ while system A of interest changes its state. The mechanism of stirring introduced in this model annuls the correlation between the system A and B at every fundamental interval τ and discards all the off-diagonal elements of the system B which acts as a reservoir.

3. DYNAMICS OF A SYSTEM OF TWO SPINS WITH NO EXTERNAL MAGNETIC FIELD AND ONE OF THE SPINS IN CONTACT WITH INFINITE TEMPERATURE BATH

Let χ and η be the wave functions of the two spins; these are then two-component vectors representing the amplitudes for spin-up and spin-down states. The complete wave function for the total system is given by

$$\psi = \chi \otimes \eta, \tag{3.1}$$

where ψ is a four-component vector. The Hamiltonian for the system is a 4×4 matrix.

Let the Hamiltonian of the combined system be given by

$$\mathcal{3C}^C = \mathcal{3C}^A + \mathcal{3C}^B + \mathcal{3C}_{int} = \mathcal{3C}_{int} = \lambda(\sigma_{1j}\sigma_{2j}),$$
$$j = 1, 2, 3 \tag{3.2}$$

where $\mathcal{3C}^A = \mathcal{3C}^B = 0$ as there is no external magnetic field acting on the system and λ denotes the strength of coupling between the two spins.

The density matrix of the combined system at $t=0$ is given by

$$\rho^C = \tfrac{1}{2}I(\tfrac{1}{2}I + p_{1j}\sigma_{1j}), \tag{3.3}$$

where

$$\rho^A = \tfrac{1}{2}I + p_{1j}\sigma_{1j} \tag{3.4}$$

and

$$\rho^B = \tfrac{1}{2}I, \tag{3.5}$$

where σ_1 and σ_2 are Pauli spin operators for systems 1 and 2, respectively, and p_{1j} ($j=1, 2, 3$) are polarizations for the spin of system 1 in direction 1, 2, and 3, respectively.

At $t=0$, system A corresponds to one of the spins which has the most general distribution and not immersed in any kind of heat bath and system B corresponds to the other spin which is immersed in an inexhaustible infinite temperature bath. At $t>0$ the two systems are brought into contact. As there is no external magnetic field applied to the system, the spins do not precess and because of the interaction between the two spins there will be relaxation alone. Since we are interested in the system A, we now calculate the relaxation of this system applying the theory developed in Sec. 2.

The unitary matrix which transforms $\rho^C(0)$ into $\rho^C(\tau)$ is given by

$$U^C(\tau) = \exp(i\lambda\sigma_{1j}\sigma_{2j}\tau)$$
$$= \exp\{\tfrac{1}{2}i\lambda\tau[(\sigma_{1j}+\sigma_{2j})^2 - \sigma_{1j}{}^2 - \sigma_{2j}{}^2]\}, \tag{3.6}$$

but we can write (3.6) as follows:

$$U^C(\tau) = \tfrac{1}{4}e^{-3i\lambda\tau}[1 + 3e^{4i\tau\lambda} + \sigma_{1j}\sigma_{2j}(e^{4i\tau\lambda}-1)], \tag{3.7}$$

where τ is the fundamental interval, and we have taken $\hbar = 1$. Now we can compute $\rho^C(1\tau)$. It is given by

$$\rho^C(1\tau) = U^C(\tau)\rho^C(0)U^{C\dagger}(\tau)$$
$$= \tfrac{1}{4} + \tfrac{1}{2}p_{1j}\sigma_{1j}\cos^2 2\tau\lambda + p_{1j}\sigma_{2j}\sin^2 2c\lambda$$
$$+ \sin 4c\lambda\, p_{1j}e_{jkl}\sigma_{1l}\sigma_{2k}. \tag{3.8}$$

If we now take the partial trace with respect to σ_2 we get

$$\rho^A(\tau) = \tfrac{1}{2}I + p_{1j}\sigma_{1j}\cos^2 2\lambda\tau. \tag{3.9}$$

In obtaining (3.9) we have used the relation

$$\mathrm{Tr}\sigma_{21} = \mathrm{Tr}\sigma_{22} = \mathrm{Tr}\sigma_{23} = 0. \tag{3.10}$$

By the method of iteration we can now calculate $\rho(n\tau)$, which is given by

$$\rho^A(n\tau) = \tfrac{1}{2}I + p_{1j}\sigma_{1j}(\cos^2 2\lambda\tau)^n. \tag{3.11}$$

If we let $n\tau = t$, (3.11) can be written as

$$\rho^A(t) = \tfrac{1}{2}I + p_{1j}\sigma_{1j}(\cos 2\lambda\tau)^{2t/\tau}. \tag{3.12}$$

The time evolution of the polarizations p_{1j} is given by

$$p_{1j}(0) \rightarrow p_{1j}(t) = (\cos 2\lambda\tau)^{2t/\tau}p_{1j}(0). \tag{3.13}$$

Now we shall compute the matrix K which maps the polarizations $p_{1j}(0)$ into $p_{1j}(t)$ and is given by

$$K = \begin{bmatrix} (\cos 2\tau\lambda)^{2t/\tau} & 0 & 0 \\ 0 & (\cos 2\tau\lambda)^{2t/\tau} & 0 \\ 0 & 0 & (\cos 2\tau\lambda)^{2t/\tau} \end{bmatrix}. \tag{3.14}$$

The matrix K is already in diagonal form, and equation (3.13) can be put in the following form:

$$p_{1j}(t) = \exp[-(2t/\tau)\ln \sec 2\tau\lambda]p_{1j}(0); \tag{3.15}$$

therefore

$$e^{-t/T} = \exp[-(2t/\tau)\ln \sec 2\tau\lambda]. \tag{3.16}$$

We see that all the three polarizations p_{11}, p_{12}, and p_{13} have the same relaxation time T given by

$$T = \tau/(2\ln \sec 2\tau\lambda). \tag{3.17}$$

It can be seen from (3.15) that, as $t \rightarrow \infty$, all three polarizations approach zero, which corresponds to an infinite temperature bath.

We can also see that there will not be any relaxation in the subsystem A when the coupling constant $\lambda = 0$. This means relaxation does not take place unless the systems A and B interact with each other and as λ increases the rate of relaxation also increases.

This model predicts that as the fundamental interval $\tau \rightarrow 0$, the relaxation in the subsystem A ceases to take place and therefore τ should be a finite nonzero quantity. Thus in this model the relaxation time T is not only a function of λ the coupling constant, but also a function of τ the fundamental interval which is true of all the models discussed in the forthcoming sections.

4. DYNAMICS OF A SYSTEM OF TWO COUPLED SPINS WITH EXTERNAL MAGNETIC FIELD IN THE z DIRECTION AND ONE OF THE SPINS IN INFINITE TEMPERATURE BATH *

The system under consideration consists of two spins. An external static magnetic field H_0 is applied in the z direction. The spin system 2 is in contact with an infinite inexhaustible temperature bath as in the previous sample and spin system 1 is assumed to have the most general distribution. The Hamiltonian of the combined system is given by

$$\mathcal{3C}^C = -\tfrac{1}{2}\gamma H_0[\sigma_{13}+\sigma_{23}]+\lambda\sigma_{1j}\sigma_{2j}, \quad j=1, 2, 3 \quad (4.1)$$

where γ is the gyromagnetic ratio, $\mathcal{3C}^A = -\tfrac{1}{2}\gamma H_0\sigma_{13}$ is the Hamiltonian of the system A, $\mathcal{3C}^B = -\tfrac{1}{2}\gamma H_0\sigma_{23}$ is the Hamiltonian of the system B, and $\mathcal{3C}_{\text{int}} = \lambda\sigma_{1j}\sigma_{2j}$ is the interaction energy of the combined system. At $t=0$ as before, system A and system B are noninteracting and the density matrix of the composite system is given by

$$\rho^C(0)=\rho^A(0)\rho^B(0), \quad (4.2)$$

where

$$\rho^A(0)=\tfrac{1}{2}I+p_{1j}\sigma_{1j} \quad (4.3)$$

and

$$\rho^B(0)=\tfrac{1}{2}I. \quad (4.4)$$

The unitary matrix which transforms $\rho^C(0)$ into $\rho^C(\tau)$ is given by

$$U^C(\tau) = \exp[-i\tau(\sigma_{13}+\sigma_{23})\gamma H_0/2+\lambda(\sigma_{1j}\sigma_{2j})]. \quad (4.5)$$

If we compute $\rho^A(\tau)$ as before, taking the partial trace with respect to system B, we obtain

$$\rho^A(\tau)=\tfrac{1}{2}I+\cos^2 2\lambda\tau[p_{1j}\sigma_{1j}\cos 2\omega\tau$$
$$-p_{1j}e_{3jk}\sigma_{1k}\sin\omega t+2p_{13}\sigma_{13}\sin^2\omega\tau], \quad (4.6)$$

where $\omega = -\gamma H_0/2$ is the precessional frequency of the spins in the external static magnetic field H_0 in the z direction. The time evolution of the polarizations p_{1j} is given by

$$p_{11}(1\tau) = \cos^2 2\lambda\tau[p_{11}(0)\cos 2\omega\tau+p_{12}(0)\sin 2\omega\tau],$$

$$p_{12}(1\tau) = \cos^2 2\lambda\tau[p_{12}(0)\cos 2\omega\tau-p_{11}(0)\sin 2\omega\tau], \quad (4.7)$$

$$p_{13}(1\tau) = \cos^2 2\lambda\tau p_{13}(0).$$

Now the matrix K which maps $p_{1j}(0)$ to $p_{1j}(1\tau)$ could be written as

$$K=\begin{bmatrix} \cos^2 2\lambda\tau\cos 2\omega\tau & -\cos^2 2\lambda\tau\sin 2\omega\tau & 0 \\ \cos^2 2\lambda\tau\sin 2\omega\tau & \cos^2 2\lambda\tau\cos 2\omega\tau & 0 \\ 0 & 0 & \cos^2 2\lambda\tau \end{bmatrix}. \quad (4.8)$$

Since the system is subjected to an external magnetic field, the matrix K given by (4.8) contains the combined effect of rotation of the polarizations in the x, y directions due to the external magnetic field in the z direction and the relaxation effect due to interaction between the two spins.

Any matrix K can be factorized into a product of unitary and Hermitian matrices,[6] i.e.,

$$K=\mathcal{3C}U. \quad (4.9)$$

In the present example the rotation of the polarizations takes place in the xy plane; the combined effect of rotation and relaxation is contained in the 3×3 matrix K and it can be factorized as follows:

$$\mathcal{3C}U=\begin{bmatrix} \cos^2 2\lambda\tau & 0 & 0 \\ 0 & \cos^2 2\lambda\tau & 0 \\ 0 & 0 & \cos^2 2\lambda\tau \end{bmatrix}\begin{bmatrix} \cos 2\omega\tau & -\sin 2\omega\tau & 0 \\ \sin 2\omega\tau & \cos 2\omega\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.10)$$

The unitary matrix is given by

$$U=\begin{bmatrix} \cos 2\omega\tau & -\sin 2\omega\tau & 0 \\ \sin 2\omega\tau & \cos 2\omega\tau & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.11)$$

and the Hermitian matrix which is responsible for the relaxation of the polarizations is given by

$$\begin{bmatrix} \cos^2 2\lambda\tau & 0 & 0 \\ 0 & \cos^2 2\lambda\tau & 0 \\ 0 & 0 & \cos^2 2\lambda\tau \end{bmatrix}. \quad (4.12)$$

Now, again by using the method of iteration, we can compute the time evolution of the polarizations corresponding to pure relaxation and this is found to be the same as in the previous example discussed in Sec. 3.

As one could have expected, the magnetic field in the z direction does not affect the rate of relaxation of the system of two coupled spins. The mechanism which causes the relaxation is the interaction between the two spins and not the external magnetic field.

5. DYNAMICS OF A SYSTEM OF TWO COUPLED SPINS WITH EXTERNAL STATIC MAGNETIC FIELD H_0 IN THE z DIRECTION AND THE RESERVOIR SPIN IN CONTACT WITH A FINITE TEMPERATURE BATH

So far we discussed a system of two coupled spins with one of the spins in contact with an infinite temperature reservoir. Now we consider spin 2 in contact with an inexhaustible finite temperature bath which would result in a Boltzmann distribution for the energy levels. Again we assume the most general distribution for spin system 1.

The Hamiltonian of the combined system is given by

$$\mathcal{3C}^C=\mathcal{3C}^A+\mathcal{3C}^B+\mathcal{3C}_{\text{int}},$$

where

$$\mathcal{3C}^A=\omega\sigma_{13}, \quad (5.1)$$

[6] F. D. Murnaghan, *The Theory of Group Representations* (The Johns Hopkins Press, Baltimore, Maryland, 1938).

$$\mathcal{3C}^B = \omega\sigma_{23}, \tag{5.2}$$

$$\mathcal{3C}_{int} = \lambda\sigma_{1j}\sigma_{2j}. \tag{5.3}$$

The density matrix of the combined system at $t=0$ is given by

$$\rho^C(0) = \rho^A(0)\rho^B(0),$$

where

$$\rho^A(0) = \tfrac{1}{2}I + p_{1j}\sigma_{1j}, \tag{5.4}$$

and

$$\rho^B(0) = \exp(-\mathcal{3C}^B/kT)/Z^B, \tag{5.5}$$
$$Z^B = \mathrm{Tr}\,\exp(-\mathcal{3C}^B/kT).$$

In the expression (5.5), k is the Boltzmann constant and T is the absolute temperature of the bath.

The unitary matrix which transforms $\rho^C(0)$ into $\rho^C(\tau)$ is given by

$$U^C(\tau) = \exp\{i[\omega(\sigma_{13}+\sigma_{23}) + \lambda\sigma_{1j}\sigma_{2j}]\tau\}, \tag{5.6}$$

where $\omega = -\gamma H_0/2$. Now we compute the density matrix for the subsystem $\rho^A(\tau)$ using (5.6) and taking the partial trace with respect to system B. It is given by

$$\rho^A(1\tau) = \tfrac{1}{2}I + \sigma_{13}[p_{13}\cos^2 2\lambda\tau - \tfrac{1}{2}\tanh x \sin^2 2\lambda\tau]$$
$$+ \sigma_{11}\{[p_{11}\cos 2\omega\tau + p_{12}\sin 2\omega\tau]\cos^2 2\tau\lambda$$
$$+ \tfrac{1}{2}\tanh x \sin 4\lambda\tau[p_{11}\sin\omega t - p_{12}\cos\omega t]\}$$
$$+ \sigma_{12}\{[p_{12}\cos 2\omega\tau - p_{11}\sin 2\omega\tau]\cos^2 2\lambda\tau$$
$$+ \tfrac{1}{2}\tanh x \sin 4\lambda\tau[p_{12}\sin 2\omega\tau + p_{11}\cos 2\omega\tau]\}, \tag{5.7}$$

where $x=\omega\beta$, $\beta=1/kT$. Since the spin systems discussed in Secs. 3 and 4 are special cases of the spin system treated here, we would like to compare the results of this section with the previous sections.

If we put $\beta=0$ and $H_0=0$ in Eq. (5.7) we obtain the following expression for $\rho^A(1\tau)$:

$$\rho^A(1\tau) = \tfrac{1}{2}I + \sigma_{13}[p_{13}\cos^2 2\lambda\tau] + \sigma_{11}[p_{11}\cos^2 2\tau\lambda]$$
$$+ \sigma_{12}[p_{12}\cos^2 2\lambda\tau]$$
$$= \tfrac{1}{2}I + p_{1j}\sigma_{1j}\cos^2 2\lambda\tau. \tag{5.8}$$

Equation (5.8) is the same as Eq. (3.9) given in Sec. 3, which corresponds to the infinite temperature bath and no external magnetic field applied to the system.

If we now just let $\beta \to 0$ in Eq. (5.7), we get an expression for $\rho^A(\tau)$ given by

$$\rho^A(1\tau) = \tfrac{1}{2}I + \sigma_{13}[p_{13}\cos^2 2\lambda\tau]$$
$$+ \sigma_{11}[p_{11}\cos 2\omega\tau + p_{12}\sin 2\omega\tau]\cos^2 2\lambda\tau$$
$$+ \sigma_{12}[p_{12}\cos 2\omega\tau - p_{11}\sin 2\omega\tau]\cos^2 2\lambda\tau. \tag{5.9}$$

The expression (5.9) is the same as Eq. (4.6) of Sec. 4 which corresponds to the infinite temperature bath and an external field H_0 applied to the system in the z direction.

The mapping $p_{1j}(\tau) = K p_{1j}(0)$ is given by

$$\begin{bmatrix} p_{11}(\tau) \\ p_{12}(\tau) \\ p_{13}(\tau) \end{bmatrix} = \begin{bmatrix} a\cos 2\omega t + b\sin 2\omega t & a\sin 2\omega t + b\cos 2\omega t & 0 \\ -a\sin 2\omega t + b\cos 2\omega t & a\cos 2\omega t + b\sin 2\omega t & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} p_{11}(0) \\ p_{12}(0) \\ p_{13}(0) - c/a \end{bmatrix}, \tag{5.10}$$

where

$$a = \cos^2 2\lambda\tau, \quad b = \tfrac{1}{2}\tanh x \sin 4\lambda\tau,$$

and

$$c = \tfrac{1}{2}\tanh x \sin^2 2\lambda\tau.$$

Again, as in the example treated in Sec. 4, here also there is the combined effect of relaxation and rotation in the x, y directions, while in the z direction there is just the pure relaxation effect. We separate these two effects by the method of polar factorization as before. The matrix K given in (5.8) can be written as

$$K = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} \cos(\theta-\delta) & \sin(\theta-\delta) & 0 \\ -\sin(\theta-\delta) & \cos(\theta-\delta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.11}$$

where $\theta = 2\omega\tau$, $\delta = \tan^{-1}(b/a)$, and $R = (a^2+b^2)^{1/2}$.

In the relation (5.11) the matrix with sines and cosines represents pure rotation. As we are interested in pure relaxation phenomena, we discuss here the pure relaxation of the polarizations p_{1j}. By using the method of iteration we find the time dependence of the polarizations p_{11} and p_{12}. We treat the case of p_{13} a little later.

$$p_{11}(n\tau) = R^n p_{11}(0) = [(a^2+b^2)^{1/2}]^n p_{11}(0), \tag{5.12}$$

$$p_{12}(n\tau) = R^n p_{12}(0) = [(a^2+b^2)^{1/2}]^n p_{12}(0). \tag{5.13}$$

Now substituting t for $n\tau$ in (5.12) and (5.13), we obtain

$$p_{11}(t) = [\cos 2\tau\lambda(\cos^2 2\tau\lambda + \tanh^2 x \sin^2 2\tau\lambda)^{1/2}]^{t/\tau} p_{11}(0), \tag{5.14}$$

$$p_{12}(t) = [\cos 2\tau\lambda(\cos^2 2\tau\lambda + \tanh^2 x \sin^2 2\tau\lambda)^{1/2}]^{t/\tau} p_{12}(0). \tag{5.15}$$

The expressions (5.14) and (5.15) can be written in the following form:

$$p_{11}(t) = e^{-t/\tau}[\ln \sec 2\tau\lambda - \tfrac{1}{2}\ln(\cos^2 2\tau\lambda$$
$$+ \tanh^2 x \sin^2 2\tau\lambda)] p_{11}(0)$$
$$= e^{-t/T_2} p_{11}(0), \tag{5.16}$$

$$p_{12}(t) = e^{-t/T_2} p_{12}(0), \tag{5.17}$$

where T_2 is the relaxation time of the polarizations p_{11} and p_{22} and is given by

$$T_2 = \frac{\tau}{\ln \sec 2\tau\lambda - \tfrac{1}{2}\ln(\cos^2 2\tau\lambda + \tanh^2 x \sin^2 2\tau\lambda)}. \tag{5.18}$$

We could immediately see from (5.16) and (5.17) that as $t \to \infty$

$$p_{11}(t) = p_{22}(t) \to \infty.$$

If we let $x \to 0$ in (5.18) the result obtained corresponds to the infinite temperature bath discussed in Secs. 3 and 4 and given by

$$T_2 = \tau/(2 \ln \sec 2\tau\lambda). \quad (5.19)$$

The expression (5.19) is the same as (3.17) given in Sec. 3.

The time evolution of the polarization in the z direction corresponds to pure relaxation as there is no rotation of the spin polarization in the z direction and is given by

$$p_{13}(\tau) = p_{13}(0) \cos^2 2\tau\lambda - \tfrac{1}{2} \tanh x \sin^2 2\tau\lambda. \quad (5.20)$$

If we calculate $p_{13}(n\tau)$ by applying the method of iteration we obtain

$$p_{13}(n\tau) = (\cos^2 2\tau\lambda)^n p_{13}(0)$$
$$- \tfrac{1}{2} \tanh x \sin^2 \tau\lambda \left[\frac{(\cos^2 2\tau\lambda)^n - 1}{\cos^2 2\tau\lambda - 1} \right]$$
$$= (\cos^2 2\tau\lambda)^n p_{13}(0)$$
$$+ \tfrac{1}{2} \tanh x [(\cos^2 2\tau\lambda)^n - 1]. \quad (5.21)$$

If we substitute $n\tau = t$ in (5.21) we get

$$p_{13}(t) = \exp[-(2t/\tau) \ln \sec^2 2\tau\lambda][p_{13}(0) + \tfrac{1}{2} \tanh x]$$
$$- \tfrac{1}{2} \tanh x \quad (5.22)$$

or

$$p_{13}(t) - p_0 = \exp[-(2t/\tau) \ln \sec^2 2\tau\lambda][p_{13}(0) - p_0]$$
$$= e^{-t/T_1}[p_{13}(0) - p_0], \quad (5.23)$$

where

$$p_0 = -\tfrac{1}{2} \tanh x$$

and

$$T_1 = \tau/(2 \ln \sec^2 2\tau\lambda). \quad (5.24)$$

If we now identify $p_{13}(t)$ with $M_z(t)$ and p_0 with M_0 we get the Bloch equation for $M_z(t)$:

$$\partial p_{13}(t)/\partial t = (2/\tau) \ln \sec^2 2\tau\lambda [p_0 - p_{13}(t)] \quad (5.25)$$

or equivalently,

$$\partial M_z(t)/\partial t = (1/T_1)[M_0 - M_z(t)]. \quad (5.26)$$

It should be noted that the relaxation times T_1 and T_2 are different in the example treated here. Further the relaxation time T_1 is the same in the spin systems treated in Secs. 3 and 4.

If we let $t \to \infty$ in Eq. (5.23), we obtain

$$p_{13}(\infty) = -\tfrac{1}{2} \tanh x. \quad (5.27)$$

The density matrix $\rho^A(t)$ for the subsystem at $t = \infty$ is given by

$$\rho^A(\infty) = \tfrac{1}{2}I - \tfrac{1}{2} \tanh x \sigma_{13}$$
$$= \frac{\cosh x - \sigma_{13} \sinh x}{2 \cosh x} = \frac{\exp(-\mathcal{H}^A\beta)}{Z^A(\beta)}, \quad (5.28)$$

since

$$\cosh x - \sigma_{13} \sinh x = e^{-\sigma_{13}x} = e^{-\sigma_{13}\beta\omega} = \exp(-\mathcal{H}^A\beta)$$

and

$$\mathrm{Tr} e^{-\sigma_{13}x} = \mathrm{Tr}(\cosh x - \sigma_{13} \sinh x) = 2 \cosh x, \quad (5.29)$$

where $\mathcal{H}^A = \omega\sigma_{13}$. The result in (5.28) shows that the system A in contact with system B (which is maintained at temperature T) attains the thermal distribution corresponding to temperature T as $t \to \infty$. Now, we write the density matrix $\rho^A(t)$. It is given by

$$\rho^A(t) = \tfrac{1}{2}I + e^{-t/T_2}p_{11}(0)\sigma_{11} + e^{-t/T_2}p_{12}(0)\sigma_{12}$$
$$+ e^{-t/T_1}p_{13}(0)\sigma_{13} + p_0\sigma_{13}(1 - e^{-t/T_1}). \quad (5.30)$$

6. DYNAMICS OF THE TWO COUPLED SPINS WHICH ARE AT EQUILIBRIUM SEPARATELY AT TEMPERATURES T_1 AND T_2, RESPECTIVELY, AT $t = 0$ AND ARE ALLOWED TO INTERACT AT $t > 0$

The system considered is a set of two spins A and B initially at equilibrium at temperatures T_1 and T_2, respectively. The system B is maintained in a thermal bath at temperature T_2 and A is not in contact with any kind of bath. The system A relaxes by virtue of its interaction with B. The problem is to study the relaxation of A and its approach to equilibrium by its interaction with B and using the stirring hypothesis with a fundamental interval τ.

The Hamiltonian for the combined system is given by

$$\mathcal{H}^C = \mathcal{H}^A + \mathcal{H}^B + \mathcal{H}_{\mathrm{int}}; \quad \mathcal{H}^A = \omega\sigma_{13}, \quad \mathcal{H}^B = \omega\sigma_{23},$$
$$\mathcal{H}^C = \omega(\sigma_{13} + \sigma_{23}) + \lambda(\sigma_{1j}\sigma_{2j}), \quad (6.1)$$

where $\omega = -\gamma H_0/2$. As in the example treated in Sec. 4, an external static magnetic field is applied to the system in the z direction.

The density matrix for the composite system at $t = 0$ is given by

$$\rho^C(0) = \rho^A(0)\rho^B(0),$$

where

$$\rho^A(0) = \exp(-\mathcal{H}^A\beta_1)/Z^A \quad (6.2)$$

and

$$\rho^B(0) = \exp(-\mathcal{H}^B\beta_2)/Z^B. \quad (6.3)$$

The quantities β_1, β_2, Z^A, and Z^B are defined as follows:

$$\beta_1 = 1/kT_1, \quad Z^A = \mathrm{Tr} \exp(-\mathcal{H}^A\beta_1),$$
$$\beta_2 = 1/kT_2, \quad Z^B = \mathrm{Tr} \exp(-\mathcal{H}^B\beta_2),$$

where k is the Boltzmann constant.

The unitary transformation which maps $\rho^C(0)$ into $\rho^C(\tau)$ is given by

$$U^C(\tau) = \exp\{-i[\omega(\sigma_{13} + \sigma_{23}) + \lambda\sigma_{1j}\sigma_{2j}]\tau\}. \quad (6.4)$$

The density matrix for the subsystem A after taking the partial trace with respect to the system B is

given by

$$\rho^A(\tau) = \frac{1}{Z^A} \left\{ \cosh\beta_1\omega - \tfrac{1}{2}\sigma_{13} \left[\cos 4\lambda\tau \frac{\sinh(\beta_1-\beta_2)\omega}{\cosh\beta_2\omega} \right. \right.$$

$$\left. \left. + \frac{\sinh(\beta_1+\beta_2)\omega}{\cosh\beta_2\omega} \right] \right\}$$

$$= \frac{1}{Z^A} \{ \cosh\beta_1\omega - \sigma_{13}[\sinh\beta_1\omega \cos^2 2\lambda\tau$$

$$+ \sin^2 2\lambda\tau \cosh\beta_1\omega \tanh\beta_2\omega] \}. \quad (6.5)$$

The density matrix $\rho^A(0)$ could be written in the following fashion:

$$\rho^A(0) = (1/Z^A)[\cosh\beta_1\omega - \sigma_{13}\sinh\beta_1\omega]. \quad (6.6)$$

If we define $\sinh\omega\beta_1 = -p_{13}(0)$ we can compute $p_{13}(1\tau)$ from (6.5). It is given by

$$p_{13}(1\tau) = \cos^2 2\lambda\tau p_{13}(0) + \sin^2 2\lambda\tau \cosh\beta_1\omega \tanh\beta_2\omega. \quad (6.7)$$

We can now calculate $p_{13}(n\tau)$ by iterating the equation above and obtain

$$p_{13}(n\tau) = (\cos^2 2\lambda\tau)^n p_{13}(0)$$

$$+ \sin^2 2\tau\lambda \cosh\omega\beta_1 \tanh\omega\beta_1 \left[\frac{(\cos^2 2\lambda\tau)^n - 1}{\cos^2 2\lambda\tau - 1} \right]$$

$$= (\cos^2 2\lambda\tau)^n p_{13}(0)$$

$$+ \cosh\omega\beta_1 \tanh\omega\beta_2[1 - (\cos^2 2\lambda\tau)^n]. \quad (6.8)$$

Now substituting t for $n\tau$ in (6.8) we get

$$p_{13}(t) = \frac{1}{Z^A(\beta_1)} \{ (\cos^2 2\lambda\tau)^{t/\tau} p_{13}(0)$$

$$+ \cosh\omega\beta_1 \tanh\omega\beta_2[1 - (\cos^2 2\lambda\tau)^{t/\tau}] \}$$

$$= \frac{1}{Z^A(\beta_1)} [(\cos^2 2\lambda\tau)^{t/\tau}(p_{13}(0) + p_0) - p_0]$$

$$= \frac{1}{Z^A(\beta_1)} [e^{-t/T_1}(p_{13}(0) + p_0) - p_0], \quad (6.9)$$

where $p_0 = -\cosh\omega\beta_1 \tanh\omega\beta_2$ and $T_1 = \tau/2 \ln \sec^2 2\tau\lambda$. The expression (6.9) is very similar to Eq. (5.23) discussed in detail in the previous section. We can now write $\rho^A(t)$ at t. It is given by

$$\rho^A(t) = \frac{1}{Z^A(\beta_1)} \{ \cosh\omega\beta_1$$

$$- \sigma_{13}[e^{-t/T_1}(p_{13}(0) - p_0) - p_0] \}. \quad (6.10)$$

We could find the equilibrium distribution of the density matrix of the subsystem A by letting $t \to \infty$ in (6.10):

$$\rho^A(\infty) = \frac{1}{Z^A(\beta_1)}(\cosh\beta_1\omega + \sigma_{13}p_0)$$

$$= \frac{1}{2\cosh\beta_1\omega}(\cosh\beta_1\omega - \sigma_{13} \cosh\beta_1\omega \tanh\omega\beta_2)$$

$$= \frac{1}{2\cosh\beta_2\omega}(\cosh\omega\beta_2 - \sigma_{13} \sinh\beta_2\omega)$$

$$= \frac{\exp(-\mathcal{3C}^A\beta_2)}{Z^A(\beta_2)}, \quad (6.11)$$

since $Z^A(\beta_1) = \mathrm{Tr} \exp(-\mathcal{3C}^A\beta_1) = 2 \cosh\beta_1\omega$, $Z^A(\beta_2) = 2 \cosh\beta_2\omega$, and $\mathcal{3C}^A = \sigma_{13}\omega$. It is seen from Eq. (6.11) that the subsystem A attains the thermal distribution corresponding to the bath at temperature T_2 as one expects. We further notice the relaxation time T_1 is the same as the ones we obtained in Secs. 3, 4, and 5, given by

$$T_1 = \tau/(2 \ln \sec^2 2\tau\lambda). \quad (6.12)$$

7. RELAXING COUPLED HARMONIC OSCILLATOR

The system considered in the example is a set of two harmonic oscillators A and B. Both the systems A and B are initially in equilibrium at temperatures T_1 and T_2 at $t=0$. At $t>0$ the system B is maintained in a thermal bath at temperature T_2, but A is not in contact with any kind of bath. System A relaxes by virtue of its interaction with system B. We study here the relaxation of system A and its approach to equilibrium by its interaction with system B using the stirring hypothesis with a fundamental interval τ.

The Hamiltonian of the combined system is given by

$$\mathcal{3C}^C = \mathcal{3C}^A + \mathcal{3C}^B + \mathcal{3C}_{int}$$

$$= [\omega a_1^\dagger a_1 + \omega a_2^\dagger a_2 + \lambda\omega(a_1^\dagger a_2 + a_2^\dagger a_1)], \quad (7.1)$$

where

$$\mathcal{3C}^A = \omega a_1^\dagger a_1, \quad \mathcal{3C}^B = \omega a_2^\dagger a_2,$$

and

$$\mathcal{3C}_{int} = \lambda\omega(a_1^\dagger a_2 + a_2^\dagger a_1).$$

We further define the following quantities:

$$a_1^\dagger = \frac{p_1 + im\omega q_1}{(2m\omega)^{1/2}}; \quad a_1 = \frac{p_1 - im\omega q_1}{(2m\omega)^{1/2}}; \quad (7.2)$$

$$a_2^\dagger = \frac{p_2 + im\omega q_2}{(2m\omega)^{1/2}}; \quad a_2 = \frac{p_2 - im\omega q_2}{(2m\omega)^{1/2}}, \quad (7.3)$$

where p_1, q_1, p_2, q_2 are momentum and position operators of the first and second harmonic oscillators, respectively. The quantity ω is the angular frequency of the harmonic oscillator; λ is the strength of coupling between the two oscillators, A and B; and m, the mass, is the same for both the oscillators.

The operators a_1, $a_1{}^\dagger$, a_2, $a_2{}^\dagger$ have the following commutation relations:

$$[a_i, a_j] = 0, \quad [a_i{}^\dagger, a_j{}^\dagger] = 0,$$
$$[a_i, a_j{}^\dagger] = \delta_{ij}, \quad i = j = 1, 2. \tag{7.4}$$

The initial density matrices $\rho^A(0)$ and $\rho^B(0)$ have the Boltzmann distribution and are given by

$$\rho^A(0) = Z_1{}^{-1}(\theta_1) e^{-a_1{}^\dagger a_1 \theta_1}, \tag{7.5}$$

$$\rho^B(0) = Z_2{}^{-1}(\theta_2) e^{-a_2{}^\dagger a_2 \theta_2}, \tag{7.6}$$

where $\theta_1 = \beta_1 \omega$, $\theta_2 = \beta_2 \omega$, $\beta_1 = 1/kT_1$, $\beta_2 = 1/kT_2$, $k =$ the Boltzmann constant, $T_1 =$ absolute temperature at which oscillator A is initially maintained, $T_2 =$ absolute temperature at which oscillator B is maintained at all times, $Z_1(\theta_1) = \mathrm{Tr} e^{-a_1{}^\dagger a_1 \theta_1}$ is the partition function for system A, and $Z_2(\theta_2) = \mathrm{Tr} e^{-a_2{}^\dagger a_2 \theta_2}$ is the partition function for system B. The density operator for the combined system ρ^C at $t = 0$ is given by

$$\rho^C(0) = \rho^A(0) \rho^B(0) = Z_1{}^{-1}(\theta_1) Z_2{}^{-1}(\theta) e^{-a_1{}^\dagger a_1 \theta_1} e^{-a_2{}^\dagger a_2 \theta_2}. \tag{7.7}$$

The unitary operator which transforms $\rho^C(0)$ into $\rho^C(\tau)$ is given by

$$U^C(\tau) = \exp(-i\mathfrak{IC}^C \tau). \tag{7.8}$$

The density matrix $\rho(\tau)$ is given by

$$\rho^C(\tau) = e^{-i\lambda(a_1{}^\dagger a_2 + a_2{}^\dagger a_1)} \rho^C(0) e^{i\lambda(a_1{}^\dagger a_2 + a_2{}^\dagger a_1)}. \tag{7.9}$$

We obtain the simple relation (7.9) since

$$[\mathfrak{IC}^A + \mathfrak{IC}^B, \mathfrak{IC}_{\mathrm{int}}] = 0$$

and

$$[\rho^C(0), \mathfrak{IC}^A + \mathfrak{IC}^B] = 0. \tag{7.10}$$

Here we have taken the density matrix in the energy diagonal representation.

Equation (7.9) could be written as

$$\rho^C(\tau) = [Z_1(\theta_1) Z_2(\theta_2)]^{-1} e^{-2ixJ_1} e^{-(\theta_1 a_1{}^\dagger a_1 + \theta_2 a_2{}^\dagger a_2)} e^{2ixJ_1}$$

$$= \frac{\exp(-2\Theta j)}{Z_1(\theta_1) Z_2(\theta_2)} e^{-2ixJ_1} \exp(\theta J_3) e^{2ixJ_1}, \tag{7.11}$$

where we have defined[7]

$$J_+ = a_1{}^\dagger a_2, \quad J_- = a_2{}^\dagger a_1, \quad J_3 = \tfrac{1}{2}(a_1{}^\dagger a_1 - a_2{}^\dagger a_2) = \tfrac{1}{2}(n_1 - n_2),$$

$$J_1 = \tfrac{1}{2}(J_+ + J_-) = \tfrac{1}{2}(a_1{}^\dagger a_2 + a_2{}^\dagger a_1),$$

$$J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2i}(a_1{}^\dagger a_2 - a_2{}^\dagger a_1),$$

$$j = \tfrac{1}{2}(a_1{}^\dagger a_1 + a_2{}^\dagger a_2) = \tfrac{1}{2}(n_1 + n_2);$$

$$m = \tfrac{1}{2}(a_1{}^\dagger a_1 - a_2{}^\dagger a_2) = J_3,$$

$$\Theta = \tfrac{1}{2}(\theta_1 + \theta_2); \quad \theta = \theta_2 - \theta_1, \quad x = \lambda \omega \tau.$$

It can easily be verified that all the J's obey angular momentum commutation rules; j and m can be considered as eigenvalues for J_2 and J_3, respectively.

We can express (7.11) as

$$\rho^C(\tau) = \frac{\exp(-2\Theta j)}{Z_1(\theta_1) Z_2(\theta_2)} e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}, \tag{7.12}$$

where

$$\cos\beta = \cos^2 p + \sin^2 p \, \cos q,$$

$$\tan\gamma = \tan\alpha = -\cos p \, \tan(q/2),$$

$$p = 2x, \quad q = -i\theta.$$

Finally, the relation between the old and the new coordinate system is given by

$$\tan\alpha = \tan\gamma = i \cos 2x \tanh(\theta/2), \tag{7.13}$$

$$\cos\beta = \cos^2 2x + \sin^2 2x \cosh\theta. \tag{7.14}$$

We obtain $\rho^A(\tau)$ by taking partial trace with respect to system B and it is given by

$$\langle n_1' | \rho^A(\tau) | n_1 \rangle = \sum_{n_2} \langle n_1 n_2 | \rho^C(\tau) | n_1 n_2 \rangle$$

$$= \delta_{n_1' n_1} \sum_{n_2} \langle n_1 n_2 | \rho^C(\tau) | n_1 n_2 \rangle \tag{7.15}$$

$$= \frac{1}{Z_1(\theta_1) Z_2(\theta_2)} \sum_{n_2} \langle n_1 n_2 | e^{-2\Theta j} e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3} | n_1 n_2 \rangle$$

$$= \frac{1}{Z_1(\theta_1) Z_2(\theta_2)} \exp\left[-2\Theta\left(\frac{n_1 + n_2}{2}\right) \right] \mathfrak{D}_{\frac{1}{2}(n_1 - n_2), \frac{1}{2}(n_1 - n_2)}^{\frac{1}{2}(n_1 + n_2)}(\alpha, \beta, \gamma), \tag{7.16}$$

where

$$\mathfrak{D}_{m', m}{}^j(\alpha, \beta, \gamma) = \sum_K (-1)^K \frac{((j+m)!(j-m)!(j+m')!(j-m')!)^{1/2}}{(j-m'-K)!(j+m-K)!K!(K+m'-m)!} e^{-i(m'\alpha + m\gamma)} (\sin\tfrac{1}{2}\beta)^{2K+m'-m} (\cos\tfrac{1}{2}\beta)^{2j+m-m'-2K},$$

[7] J. Schwinger, Notes on Angular Momentum, NYO-3071, 1952.

$j = (n_1+n_2)/2$, $m = (n_1-n_2)/2$, and K is an integer. Now we can write (7.15) as

$$\langle n_1' | \rho^A(\tau) | n_1 \rangle \delta_{nn'} = \frac{1}{Z_1(\theta_1)Z_2(\theta_2)} \sum_{n_2} \exp[-\Theta(n_1+n_2)] \exp[-\tfrac{1}{2}i(n_1-n_2)(\alpha+\gamma)]$$

$$\times \sum_K (-1)^K \frac{n_1! n_2!}{(n_2-K)!(n_1-K)!K!}(\cos\tfrac{1}{2}\beta)^{n_1+n_2-2K}(\sin\tfrac{1}{2}\beta)^{2K}. \quad (7.16a)$$

We finally obtain the following result after summing over n_2 and K:

$$\langle n_1' | \rho^A(\tau) | n_1 \rangle = \frac{\delta_{nn'}}{Z_1(\theta_1)Z_2(\theta_2)} \left[\left(\frac{\cos\tfrac{1}{2}\beta}{\exp[\Theta+(i/2)(\alpha+\gamma)]} \right) \left(\frac{1-\sec\tfrac{1}{2}\beta \exp[i(\gamma+\alpha/2)-\Theta]}{1-\cos\tfrac{1}{2}\beta \exp[i(\gamma+\alpha/2)-\Theta]} \right) \right]^{n_1}$$

$$\times \{1-\cos\tfrac{1}{2}\beta \exp[i(\gamma+\alpha/2)-\Theta]\}^{-1}. \quad (7.17)$$

If we now substitute $\alpha = \gamma = i\delta$ in (7.13) we get

$$\tanh\delta = \cos 2x \tanh(\theta/2). \quad (7.18)$$

Now substituting (7.18) in (7.17) we finally get

$$Z_1(\theta_1)Z_2(\theta_2)\langle n_1' | \rho^A(\tau) | n_1 \rangle = \delta_{n_1'n_1} \left[\frac{\cos\tfrac{1}{2}\beta \exp(\delta+\Theta)-1}{\exp(\Theta-\delta)[\exp(\Theta+\delta)-\cos\tfrac{1}{2}\beta]} \right]^{n_1} \frac{1}{1-\cos\tfrac{1}{2}\beta e^{-(\Theta+\delta)}}. \quad (7.19)$$

It could be shown that the trace condition $\mathrm{Tr}\rho^A(\tau)=1$ is satisfied. Equation (7.19) could be written as

$$\langle n_1' | \rho^A(\tau) | n_1 \rangle = \delta_{n_1'n_1} Z(\theta_1(\tau)) e^{-n_1\theta_1(\tau)} \quad (7.20)$$

with

$$e^{-\theta_1(\tau)} = \left(\frac{\cos(\tfrac{1}{2}\beta) \exp(\Theta+\delta)-1}{\exp(\Theta-\delta) \exp(\Theta+\delta)-\cos\tfrac{1}{2}\beta} \right). \quad (7.21)$$

The quantity $\theta_1(\tau)$ thus replaces $\theta_1(0) = \Theta - \theta/2$ as the effective inverse temperature after stirring once at the instant τ. It could be seen from (7.20) that the initial Boltzmann distribution (7.5) attains another Boltzmann distribution at temperature $[\theta_1(\tau)]^{-1}$ at the first fundamental interval τ. Now we can compute $[\theta_1(2\tau)]^{-1}$, $[\theta_1(3\tau)]^{-1}$, $\cdots[\theta_1(n\tau)]^{-1}$ to show that the system A attains the same temperature as that of system B as $n\tau = t \rightarrow \infty$. Unfortunately, there is no simple analytical method of carrying out the iteration

CURVE A: $\theta_1 = 10$, $\theta_2 = 5$, $T_1 = \tfrac{1}{2}T_2$
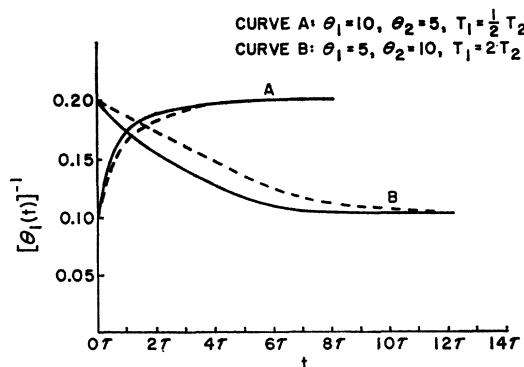CURVE B: $\theta_1 = 5$, $\theta_2 = 10$, $T_1 = 2T_2$

FIG. 1. The "temperature" $Jk/h\nu = [\theta_1(t)]^{-1}$ as a function of time t for the relaxation of an initial Boltzmann distribution. Broken curves refer to Montroll and Shuler and solid lines refer to the present results.

process as in the case of spin systems. Since the transient distribution of the relaxing oscillator is always canonical in this case it is possible to characterize it by a "temperature" $J(t) = h\nu/k\theta(t)$. This result is similar to the result obtained by Montroll and Shuler,[8] and Mathews, Shapiro, and Falkoff[9] in the sense that the transient distribution of the relaxing oscillator is always canonically distributed. Even though it is not possible to compare the analytical expressions for $\theta_1(\tau)$ as in our case there is no simple method of iterating the expression (7.21), we have shown in Fig. 1 that Montroll and Schuler's graphical results agree with the results obtained here. It could be seen in Fig. 1 that the relaxing oscillator attains the same temperature T_2 of the bath as $t \rightarrow \infty$.

ACKNOWLEDGMENTS

[8] E. W. Montroll and K. E. Shuler, J. Chem. Phys. 26, 454 (1957).
[9] P. M. Mathews, I. I. Shapiro, and D. L. Falkoff, Phys. Rev. 120, 1 (1960).